Clex 2: The rules of probability

Problem 1: Joint Distributions and the Fundamental Rules

Consider two discrete random variables, $X \in \{0, 1\}$ and $Y \in \{0, 1, 2\}$, with the joint probability mass function $P(X, Y)$ defined by the following table:

$P(X,Y)$	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	0.10	0.15	0.05
$X = 1$	0.20	0.30	0.20

a) Use the **Sum Rule** (Marginalization) to calculate the marginal distributions $P(X)$ and $P(Y)$. Verify that $\sum_x P(X = x) = 1$.

b) Calculate the conditional probability $P(Y|X = 0)$ for all values of Y.

c) Calculate the conditional expected value $E(Y|X = 0)$.

d) Demonstrate the **Product Rule** by showing that $P(X = 0, Y = 1) = P(Y = 1|X = 0)P(X = 0)$.

Problem 2: Bayesian Inference and Terminology

A robotic vacuum cleaner uses a laser sensor to detect if it is in a "Clear" (C) or "Obstacle" (O) state. Based on previous navigation data, the robot has a **prior** belief that there is a 20% chance it is heading toward an obstacle ($P(O) = 0.2$). The sensor's performance is characterized by the following **likelihoods**:

- If there is an obstacle, the sensor correctly returns a "Red" signal (R) with probability $P(R|O) = 0.9$.
- If the path is clear, the sensor erroneously returns a "Red" signal with probability $P(R|C) = 0.15$.

a) Identify and state the numerical values for the **Prior** and the **Likelihood** of a "Red" signal given an obstacle.

b) Calculate the **Evidence** (marginal likelihood) $P(R)$ using the Law of Total Probability.

c) If the sensor flashes "Red", use Bayes' Theorem to calculate the **Posterior** probability that there is actually an obstacle, $P(O|R)$.

d) Compare the posterior $P(O|R)$ to the prior $P(O)$. Qualitatively explain how the "Red" signal updated the robot's belief.

Problem 3: Frequentist vs. Bayesian Perspectives

A box contains two types of coins:

- **Type F**: A Fair coin ($P(H) = 0.5$).
- **Type B**: A Biased coin ($P(H) = 0.9$).

Suppose the box has 9 Fair coins and 1 Biased coin. You pick one coin at random and flip it twice, resulting in two Heads (HH).

a) **Frequentist View:** Calculate the Likelihood of the data $P(HH|\text{type})$ for both types of coins. Based strictly on which model makes the observed data most likely, which coin would a Frequentist prefer?

b) **Bayesian View:** Using the distribution of coins in the box as your **Prior** ($P(F) = 0.9, P(B) = 0.1$), calculate the **Posterior** probability that the coin you picked is actually the Biased one, $P(B|HH)$.

c) **Comparison:** Does the Bayesian believe the coin is biased? Why is this different from the Frequentist conclusion?