Clex 3: Discrete Random Variables

Problem 1: Sensor reliability

A network administrator monitors the number of connectivity drops X experienced by an IoT gateway over a 24-hour period. The probability distribution is as follows:

x	0	1	2	3	4	5
$P(X=x)$	0.45	0.25	0.15	0.08	0.05	0.02

1. Find the expected number of drops $E[X]$.

   Solution: $E[X] = \sum_i x_i P(X = x_i)$

   $$E[X] = (0\times 0.45)+(1\times 0.25)+(2\times 0.15)+(3\times 0.08)+(4\times 0.05)+(5\times 0.02)$$

   $$E[X] = 0 + 0.25 + 0.30 + 0.24 + 0.20 + 0.10 = 1.09$$

2. Calculate the variance $\text{Var}(X)$.

   Solution: First, find $E[X^2] = \sum_i x_i^2 P(X = x_i)$:

   $$E[X^2] = (0^2\times 0.45)+(1^2\times 0.25)+(2^2\times 0.15)+(3^2\times 0.08)+(4^2\times 0.05)+(5^2\times 0.02)$$

   $$E[X^2] = 0 + 0.25 + 0.60 + 0.72 + 0.80 + 0.50 = 2.87$$

   Using $\text{Var}(X) = E[X^2] - (E[X])^2$:

   $$\text{Var}(X) = 2.87 - (1.09)^2 = 2.87 - 1.1881 = 1.6819$$

Problem 2: Optical lenses

A manufacturing process for optical lenses has a defect rate of $p = 0.0015$. A batch of $n = 2000$ lenses is produced.

1. Identify the parameters for the Binomial distribution modeling the number of defects X.

   Solution: The parameters are $n = 2000$ and $p = 0.0015$.

2. Define the parameter λ for a Poisson approximation.

   Solution: $\lambda = np = 2000 \times 0.0015 = 3$.

3. Approximate $P(X = 3)$ using the Poisson formula: $P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$.

   Solution:

   $$P(X = 3) \approx \frac{3^3 e^{-3}}{3!} = \frac{27e^{-3}}{6} = 4.5e^{-3} \approx 4.5 \times 0.049787 = 0.22404$$

Problem 3: API requests

Context: A cloud service provider classifies incoming API requests into four distinct types:

- C_1: Read requests ($p_1 = 0.60$)
- C_2: Write requests ($p_2 = 0.20$)
- C_3: Delete requests ($p_3 = 0.14$)
- C_4: Admin requests ($p_4 = 0.06$)

Let $\mathbf{X} = (X_1, X_2, X_3, X_4) \sim \text{Multinomial}(n = 27, \{p_1, p_2, p_3, p_4\})$.

1. **Marginal Distribution:** Prove that the marginal distribution of X_1 is binomial. What are the parameters of this binomial distribution?

   Solution: In a Multinomial trial, we can define a "success" as a request falling into C_1 (with probability p_1) and a "failure" as falling into any other category (with probability $1 - p_1$). Since the n trials are independent and the probability p_1 is constant, X_1 follows a Binomial distribution. Parameters: $n = 27, p = 0.60$.

2. **Conditional Distribution:** Prove that the conditional probability mass function $P(X_2, X_3, X_4 \mid X_1 = 18)$ corresponds to a multinomial distribution. What are the parameters of this multinomial distribution?

   Solution: Given $X_1 = 18$, there are $n' = n - 18 = 27 - 18 = 9$ remaining requests to be distributed among C_2, C_3, and C_4. The relative probabilities for these categories are $p_i' = \frac{p_i}{1-p_1}$. Parameters: $n' = 9$ and probabilities:

$$p_2' = \frac{0.20}{1 - 0.60} = 0.50, \quad p_3' = \frac{0.14}{0.40} = 0.35, \quad p_4' = \frac{0.06}{0.40} = 0.15$$

3. Compute $P(X_2 = 5, X_3 = 3, X_4 = 1 \mid X_1 = 18)$

   Solution: Using the Multinomial PMF with $n' = 9$ and $k = \{5, 3, 1\}$:

$$P = \frac{9!}{5!3!1!}(0.50)^5(0.35)^3(0.15)^1 = 0.10129$$