Clex 5: Partial derivatives - Solutions

Problem 1: Just derivatives

Calculate all first and all second partial derivatives of the function

$$f(x,y,z) = x^2 - 2\sqrt{y^2+z^2}$$

Solution:

First, we find the first-order partial derivatives:

- $f_x = \frac{\partial}{\partial x}(x^2 - 2(y^2+z^2)^{1/2}) = 2x$
- $f_y = -2 \cdot \frac{1}{2}(y^2+z^2)^{-1/2} \cdot 2y = \frac{-2y}{\sqrt{y^2+z^2}}$
- $f_z = -2 \cdot \frac{1}{2}(y^2+z^2)^{-1/2} \cdot 2z = \frac{-2z}{\sqrt{y^2+z^2}}$

Next, we calculate the second-order partial derivatives:

- $f_{xx} = \frac{\partial}{\partial x}(2x) = 2$
- $f_{xy} = f_{xz} = f_{yx} = f_{zx} = 0$
- $f_{yy} = \frac{\partial}{\partial y}\left[-2y(y^2+z^2)^{-1/2}\right] = -2(y^2+z^2)^{-1/2} + (-2y)(-\frac{1}{2})(y^2+z^2)^{-3/2}(2y) = \frac{-2z^2}{(y^2+z^2)^{3/2}}$
- $f_{zz} = \frac{\partial}{\partial z}\left[-2z(y^2+z^2)^{-1/2}\right] = -2(y^2+z^2)^{-1/2} + (-2z)(-\frac{1}{2})(y^2+z^2)^{-3/2}(2z) = \frac{-2y^2}{(y^2+z^2)^{3/2}}$
- $f_{yz} = f_{zy} = \frac{\partial}{\partial z}\left[-2y(y^2+z^2)^{-1/2}\right] = -2y(-\frac{1}{2})(y^2+z^2)^{-3/2}(2z) = \frac{2yz}{(y^2+z^2)^{3/2}}$

Problem 2: Tangent hyperplane

Determine the the tangent hyperplane and the normal vector to the graph of the function

$$f(x,y,z) = xy^2 + y\cos(\pi(z-1))$$

at the point $(x,y,z) = (2,1,2)$.

Solution:

Let $w = f(x,y,z)$. The point on the graph in $\mathbb{R}^4$ is $(2,1,2,f(2,1,2))$.

$$f(2,1,2) = 2(1)^2 + 1 \cdot \cos(\pi(2-1)) = 2 + \cos(\pi) = 2 - 1 = 1$$

We find the partial derivatives f at $(2,1,2)$:

- $f_x = y^2 \implies f_x(2,1,2) = 1$
- $f_y = 2xy + \cos(\pi(z-1)) \implies f_y(2,1,2) = 2(2)(1) + \cos(\pi) = 4 - 1 = 3$

- $f_z = -y\pi\sin(\pi(z-1)) \implies f_z(2,1,2) = -(1)\pi\sin(\pi) = 0$

The **normal vector** to the graph in $\mathbb{R}^4$ is $\vec{n} = \langle f_x, f_y, f_z, -1\rangle = \langle \mathbf{1}, \mathbf{3}, \mathbf{0}, -\mathbf{1}\rangle$. The equation of the **tangent hyperplane** is:

$$1(x-2) + 3(y-1) + 0(z-2) - 1(w-1) = 0 \implies \mathbf{w} = \mathbf{x} + \mathbf{3y} - \mathbf{4}$$

Problem 3: Horizontal tangent plane

Determine all horizontal tangent planes for the graph of the function

$$z = x^3 + 3xy + y^3$$

Solution:

A tangent plane is horizontal where the partial derivatives f_x and f_y are both zero:

1. $f_x = 3x^2 + 3y = 0 \implies y = -x^2$
2. $f_y = 3x + 3y^2 = 0 \implies x + y^2 = 0$

Substituting $y = -x^2$ into the second equation:

$$x + (-x^2)^2 = 0 \implies x + x^4 = 0 \implies x(1 + x^3) = 0$$

This gives two critical points:

- If $x = 0$, then $y = 0$. The height is $z(0,0) = 0$. The plane is $z = 0$.
- If $x = -1$, then $y = -(-1)^2 = -1$. The height is $z(-1,-1) = (-1)^3 + 3(-1)(-1) + (-1)^3 = -1 + 3 - 1 = 1$. The plane is $z = 1$.