Clex 6: Multivariate Chain Rules - Solutions

Problem 1: One independent variable

Using chain rule determine the derivative dw/dt for the function

$$w = \ln(x^2 + y^2 + z^2),\ x = \cos(t),\ y = \sin(t),\ z = 4\sqrt{t}$$

at the point $t = 3$.

Solution:

By the multivariate chain rule for one independent variable:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

1. **Calculate the partial derivatives of w:**

$$\frac{\partial w}{\partial x} = \frac{2x}{x^2 + y^2 + z^2}, \quad \frac{\partial w}{\partial y} = \frac{2y}{x^2 + y^2 + z^2}, \quad \frac{\partial w}{\partial z} = \frac{2z}{x^2 + y^2 + z^2}$$

2. **Calculate the derivatives of the inner functions:**

$$\frac{dx}{dt} = -\sin(t), \quad \frac{dy}{dt} = \cos(t), \quad \frac{dz}{dt} = \frac{2}{\sqrt{t}}$$

3. **Evaluate at $t = 3$:**

- $x = \cos(3)$, $y = \sin(3)$, $z = 4\sqrt{3}$
- $x^2 + y^2 + z^2 = \cos^2(3) + \sin^2(3) + (4\sqrt{3})^2 = 1 + 48 = 49$

4. **Substitute into the chain rule:**

$$\frac{dw}{dt} = \frac{2\cos(3)(-\sin 3) + 2\sin(3)(\cos 3) + 2(4\sqrt{3})\left(\frac{2}{\sqrt{3}}\right)}{49}$$

$$\frac{dw}{dt} = \frac{-2\sin 3\cos 3 + 2\sin 3\cos 3 + 16}{49} = \frac{16}{49}$$

Problem 2: Three independent variables

Using chain rule determine the partial derivatives $\partial u/\partial x$, $\partial u/\partial y$ and $\partial u/\partial z$ for the function

$$u = \frac{p - q}{q - r},\ p = x + y + z,\ q = x - y + z,\ r = x + y - z$$

at the point $(x, y, z) = (\sqrt{3}, 2, 1)$.

Solution:

At the point $(\sqrt{3}, 2, 1)$, we have $p = 3 + \sqrt{3}$, $q = \sqrt{3} - 1$, and $r = \sqrt{3} + 1$. This gives $p - q = 4$ and $q - r = -2$.

1. **Partial derivatives of u with respect to p, q, r:**

$$\frac{\partial u}{\partial p} = \frac{1}{q - r} = -\frac{1}{2}, \quad \frac{\partial u}{\partial r} = \frac{p - q}{(q - r)^2} = \frac{4}{(-2)^2} = 1$$

$$\frac{\partial u}{\partial q} = \frac{-(q - r) - (p - q)}{(q - r)^2} = \frac{r - p}{(q - r)^2} = \frac{-2}{4} = -\frac{1}{2}$$

2. **Partial derivatives of p, q, r with respect to x, y, z:**

$$\begin{array}{lll} p_x = 1 & p_y = 1 & p_z = 1 \\ q_x = 1 & q_y = -1 & q_z = 1 \\ r_x = 1 & r_y = 1 & r_z = -1 \end{array}$$

3. **Apply the Chain Rule:**

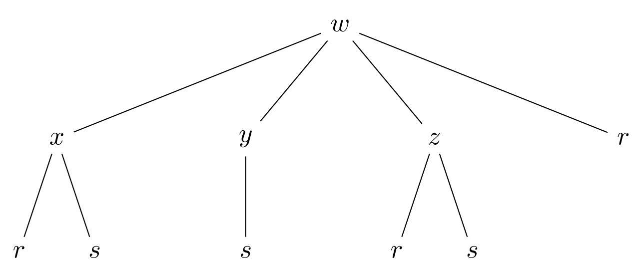
- $\frac{\partial u}{\partial x} = u_p p_x + u_q q_x + u_r r_x = (-\frac{1}{2})(1) + (-\frac{1}{2})(1) + (1)(1) = 0$
- $\frac{\partial u}{\partial y} = u_p p_y + u_q q_y + u_r r_y = (-\frac{1}{2})(1) + (-\frac{1}{2})(-1) + (1)(1) = 1$
- $\frac{\partial u}{\partial z} = u_p p_z + u_q q_z + u_r r_z = (-\frac{1}{2})(1) + (-\frac{1}{2})(1) + (1)(-1) = -2$

Problem 3: Tree diagrams

Draw a tree diagram and write the chain rule for the derivatives $\partial w/\partial r$ and $\partial w/\partial s$ where the function w is specified by

$$w = f(x, y, z, r), \; x = g(r, s), \; y = h(s), \; z = k(r, s)$$

Solution:



The chain rule formulas (following the paths to each independent variable) are:

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r} + \frac{\partial f}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$