Homework 3

Math for AI

Instructions: Please solve the following problems showing all your work. State any assumptions you make.

1. Find the first-order partial derivatives f_x and f_y for the function:

   $$f(x, y) = x^4y^3 - e^{2x}\sin(y) + \ln(x^2 + 1)$$

   at the point $(1, 0)$.

2. Find the equation of the tangent plane to the surface $z = x^2+3xy+y^2$ at the point $(x, y, z) = (1, -2, -1)$.

3. Find the equation of the tangent hyperplane to the surface $w = \sqrt{20 - x^2 - 3y^2 - 4z^2}$ at the point $(x, y, z, w) = (2, 1, -1, 3)$.

4. Find all points (x, y, z, u) on the surface $u = x^3-3xz+y^2-4yz$ where the tangent hyperplane is horizontal.

5. Let $f(x, y, z) = e^{-z}\sin(x^2y)$. Compute all second-order partial derivatives of f.

6. Verify Clairaut's Theorem (show that $f_{xy} = f_{yx}$) for the function:

   $$f(x, y) = x^2e^{xy}$$

7. Let $z = x^2y + xy^2$, where $x = \cos(t)$ and $y = \sin(t)$. Use the Chain Rule to find $\frac{dz}{dt}$ as a function of t.

8. Suppose $w = \ln(x + 2y + z^2)$, where $x = e^t$, $y = t^2$, and $z = t$. Find the value of $\frac{dw}{dt}$ when $t = 1$.

9. Let $z = f(x, y) = x^2 - y^2$, where $x = u - v$ and $y = u + v$. Use the Chain Rule to find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ in terms of u and v.

10. Let $w = g(x, y, z)$ be a differentiable function, and let x, y, z be defined in terms of spherical coordinates:

    $$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

    Write out the Chain Rule formula for $\frac{\partial w}{\partial \phi}$. (Note: Express your answer in terms of the partial derivatives of g and the trigonometric functions of ρ, ϕ, θ).

11. The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of an ideal gas are related by the equation $PV = 8.31T$. Suppose the temperature is increasing at a rate of $2\,\text{K/s}$ and the pressure is increasing at a rate of $0.5\,\text{kPa/s}$. Find the rate of change of the volume when $T = 300\,\text{K}$ and $P = 100\,\text{kPa}$.

12. The radius of a right circular cone is increasing at a rate of 1.5 cm/s while its height is decreasing at a rate of 2.5 cm/s. At what rate is the volume of the cone changing when the radius is 120 cm and the height is 140 cm? (Recall that $V = \frac{1}{3}\pi r^2 h$).