Math for AI - Homework 4

1. Let $f(x, y, z) = x^2y + z\ln(x)$.
   (a) Compute the gradient vector $\nabla f(x, y, z)$.
   (b) Find the equation of the tangent hyperplane to the level surface $f(x, y, z) = 2$ at the point $(1, 2, 0)$.

2. Calculate the directional derivative of the function $g(x, y) = e^x \cos(y)$ at the point $(0, \frac{\pi}{4})$ in the direction of the vector $\mathbf{v} = \langle 3, -4\rangle$.

3. Consider the function $h(x, y, z) = x^2 + y^2 - z^2$.
   (a) In what direction does h increase most rapidly at the point $(1, 1, 1)$? Express your answer as a unit vector.
   (b) What is the maximum rate of change of h at this point?

4. Consider the cost function $J(w_1, w_2) = w_1^2 + 2w_2^2 - 2w_1w_2 + 2w_2$. Perform two iteration of gradient descent starting from the initial point $\mathbf{w}^{(0)} = (0, 0)$ with a learning rate of $\alpha = 0.1$. What are the descent points $\mathbf{w}^{(1)}$ and $\mathbf{w}^{(2)}$?

5. Let $f(x, y) = \sqrt{x^2 + y^3}$. Compute the total differential df at the point $(1, 2)$. Use this linear approximation to estimate the value of $f(1.02, 1.97)$.

6. Find the Jacobian matrix $\mathbf{J}$ for the transformation $\mathbf{F}(u, v) = \begin{bmatrix} u^2\cos(v) \\ u\sin(v) \end{bmatrix}$. Evaluate $\mathbf{J}$ at the point $(u, v) = (2, \frac{\pi}{2})$.

7. Let $\mathbf{y} = \mathbf{f}(\mathbf{x})$ be defined by $y_1 = x_1^2 - x_2^2$ and $y_2 = 2x_1x_2$. Let $z = g(\mathbf{y}) = y_1^2 + y_2^2$. Compute the Jacobian matrices $\mathbf{J}_\mathbf{f}(\mathbf{x})$ and $\mathbf{J}_g(\mathbf{y})$. Then, using the multivariable Chain Rule for compositions of Jacobians, find the gradient of the composite function $z(\mathbf{x})$ evaluated at $\mathbf{x} = (1, 1)$.

8. Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{W} \in \mathbb{R}^{m\times n}$. Define the transformation $\mathbf{z} = \mathbf{W}\mathbf{x}$ and $\mathbf{h} = \sigma(\mathbf{z})$, where $\sigma : \mathbb{R} \to \mathbb{R}$ is an activation function applied element-wise. Derive the Jacobian matrix $\frac{\partial \mathbf{h}}{\partial \mathbf{x}}$ in terms of $\mathbf{W}$ and the derivative of the activation function, σ'.

9. Find and classify all critical points of the function $f(x, y) = x^3 - 3x + y^3 - 3y^2$.

10. Find and classify all critical points of the function $f(x, y) = \frac{1}{x} + xy + \frac{1}{y}$.

11. Use the method of Lagrange multipliers to find the absolute maximum and minimum values of $f(x, y) = xy$ subject to the equality constraint $x^2 + y^2 = 1$.

12. Use the method of Lagrange multipliers to find the point on the plane $x + 2y - z = 6$ that is closest to the origin.
(Hint: It is computationally easier to minimize the squared distance function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the given constraint).