# Math for AI - Practice Test 1

**Instructions:** Show all your work. Justify your steps where appropriate.

**Problem 1.** Professor Max gives a test in his probability class. He knows from historical data that 55% of his students regularly do the assigned homework without AI help. If a student does the homework without AI help, they have a 96% chance of passing the test. If they do use AI to solve the homework for them, they only have a 18% chance of passing.

(a) What is the overall probability that a randomly selected student passes the test?

(b) Given that a randomly selected student passed the test, what is the probability that they actually did the assigned homework without AI help?

(c) Are passing the test and using AI help for the homework independent. Justify you answer numerically.

**Problem 2.** An AI classification model categorizes incoming network requests into three classes: "Safe", "Warning", and "Malicious". This represents a categorical distribution where the probabilities are $p_{\text{safe}} = 0.95$, $p_{\text{warning}} = 0.04$, and $p_{\text{malicious}} = 0.01$. Assume requests are independent.

(a) If a server receives $n = 5$ requests, what is the exact probability that 3 are Safe, 1 is Warning, and 1 is Malicious?

(b) Let $X$ be the number of Malicious requests out of $n = 20$ requests. Identify the exact distribution of $X$ and write the expression for $P(X = 2)$.

(c) If the server receives a large batch of $n = 300$ requests, use the Poisson approximation to estimate the probability of receiving exactly 4 Malicious requests.

**Problem 3.** Consider a continuous random variable $X$ defined on a finite interval with the probability density function (PDF):

$$f(x) = \begin{cases} c(1 - x^2) & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the constant $c$ that makes $f(x)$ a valid PDF.

(b) Find the cumulative distribution function (CDF), $F(x)$, for $x \in [-1, 1]$.

(c) Compute the expected value $\mathbb{E}[X]$ and the variance $\text{Var}(X)$.

(d) Suppose we draw a sample of $n = 100$ independent observations from this distribution to train a small model. Using the Central Limit Theorem, approximate the probability that the sample mean $\bar{X}$ is greater than 0.05. (You may leave your answer in terms of the standard normal CDF, $\Phi$).

**Problem 4.** Consider the 3D graph (surface) defined by the function of two variables $z = f(x, y) = x^2 \ln(y) + ye^{x-1}$. Find the equation of the tangent plane (hyperplane in $\mathbb{R}^3$) to this graph at the point $(x, y) = (1, 1)$.

**Problem 5.** In physics-informed neural networks, we often require functions to satisfy a differential equation exactly. Let $u(x,t) = e^{-\alpha^2 k^2 t}\sin(kx)$, where $\alpha$ and $k$ are constants. Compute the necessary partial derivatives to show that $u(x,t)$ satisfies the one-dimensional heat equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

**Problem 6.** Let $z = f(x,y) = x^2y - y^3$. Suppose $x$ and $y$ are themselves functions of two independent variables $u$ and $v$, given by $x = u^2 + v$ and $y = u - v^2$.

(a) Draw the dependency tree diagram for this system.

(b) Use the multivariable chain rule to compute $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$. Evaluate both derivatives at the point $(u,v) = (1,1)$.

**Problem 7.** Consider the loss surface defined by $L(w_1, w_2) = 3w_1^2 - 2w_1w_2 + w_2^2$. Let the current weights be at the point $P(1,-1)$.

(a) Find the gradient vector $\nabla L$ at $P$.

(b) What is the maximum rate of increase of $L$ at $P$, and in what unit direction does it occur?

(c) Compute the directional derivative of $L$ at $P$ in the direction of the vector $\mathbf{v} = \langle 3, 4\rangle$.

**Problem 8.** Consider the polar coordinate transformation $T : (r,\theta) \mapsto (x,y)$ given by $x = r\cos\theta$ and $y = r\sin\theta$. Let $S : (x,y) \mapsto (u,v)$ be a mapping given by $u = x^2 - y^2$ and $v = 2xy$.

(a) Compute the Jacobian of $T$, $\frac{\partial(x,y)}{\partial(r,\theta)}$ at the point $(r,\theta) = (2,\pi/4)$.

(b) Compute the Jacobian of $S$, $\frac{\partial(u,v)}{\partial(x,y)}$ at the point $(x,y) = (\sqrt{2},\sqrt{2})$.

(c) Check that the composite transformation $S \circ T$ has a Jacobian at the point $(r,\theta) = (2,\pi/4)$ which is the matrix product of the two previously computed Jacobians.

**Problem 9.** Find all the critical points of the function $f(x,y) = x^3 - 3x + y^3 - 3y^2$. Use the Second Partials Test to classify each critical point as a local minimum, local maximum, or saddle point.

**Problem 10.** An AI agent needs to allocate three computational resources $x, y$, and $z$ to maximize a linear utility function $U(x,y,z) = 2x + 3y + z$. However, the resources are bounded by the spherical constraint $x^2 + y^2 + z^2 = 14$. Use the method of Lagrange multipliers to find the points $(x,y,z)$ where the absolute maximum and minimum of $U$ occur, and state these maximum and minimum values.